A Method for the Calculation of the Radiation-Pattern and Mode-Conversion Properties of a Solid-State Heterojunction Laser

LEONARD LEWIN, SENIOR MEMBER, IEEE

***Abstract*—A solution of the laser fields, both inside and outside the laser, is given in terms of the mode-conversion coefficients and an integral equation for the radiation pattern. It is shown how very accurate analytic solutions can be obtained by what, at first sight, appear to be extremely crude approximations. The reason is that mode conversion is taken *implicitly* into account by using a multiplier, whose exact form does not appear to be very critical, as a weighting function to average two different formulas for the function representing the radiation; and with the correct form for it, *all* the mode-conversion and reflection coefficients can be legitimately ignored. A plane-wave formula for this multiplier is a good first approximation, and a number of existing expressions occurring in the literature are obtained in this way. It is also shown rigorously that the results of an earlier obliquity-factor analysis apply. Further refinements are introduced to allow for higher order discrete modes, and good approximate analytic forms for the mode-reflection and conversion coefficients are obtained. A check with a rather extreme example shows excellent agreement with Ikegami's numerical computation for the dominant-mode reflection at the laser–air interface.**

The methods of this paper are applicable to general laser structures of cylindrical geometry with either continuous or discontinuous variations in refractive index. Very accurate numerical solutions should be obtainable after only one iteration of the integral equation, starting with the reflection-modified form of Hockham's formula as initiating function.

I. INTRODUCTION

THE SIMPLEST formula for the radiation pattern involves a diffraction-type calculation based on the field distribution of an exciting mode. Casey, Panish, and Merz [1] have shown that this leads to a fairly good agreement with experiments, but that the measured radiation pattern is somewhat sharper than the theoretical one. Thompson [2] included an empirical *intensity* factor cos θ (where θ is the angle from the axis to the field point), and Hockham [3] showed that a *field-strength* factor cos θ is required, along with some other factors whose effect is relatively slight. Lewin [4] showed that Hockham's formula could be interpreted as the Huygens' obliquity factor for the arrangement, and suggested a minor correction to it to take explicit account of the reflection of the exciting mode at the air–laser interface. Both Hockham's formula, and its modification, give excellent agreement with measurements, though there is now a just discernible tendency [3], [5] to underestimate the beamwidth. The effect is very slight, and is of the order of the experimental and measurement errors, but seems sufficiently consistent to suggest that the discrepancy is a real one. Butler and Zoroofchi [6] produce a formula which does not differ substantially from Hockham's, so far as numerical results are concerned, and they fit theoretical patterns to experimental profiles from several devices to determine the model geometry and compare the results with known dielectric parameters of the device structure.

All these calculations use the form of an incident field in the calculations, and imply that mode conversion is sufficiently small to be neglected. Ikegami [7] estimates this effect as less than 0.5 percent, and Butler and Zoroofchi claim that mode conversion is inconsequential. They find that the values of dielectric steps and layer thicknesses derived from their curve-fitting process agree "reasonably well" with values deduced from analysis of the material and the processing [6].

The purpose of the present paper is to refine the analysis by taking mode conversion explicitly into account and to verify the accuracy of the currently used formulas. This should give confidence to the parameter values found from curve-fitting, and permit consideration of any remaining discrepancies, should any be found, to be related to possible effects not so far allowed for in the theory, such as finite stripe breadth, or nonlinear or dynamic processes in the lasing material. It is in fact found that Hockham's formula, with the mode-reflection modification, is already very accurate, and implicitly takes some mode conversion into account. Along with its use, self-consistent values of reflection and mode-conversion coefficients can be calculated.

II. THE LASER FIELD

We assume that the junction (see Fig. 1) is excited by the dominant mode in the transverse electric configuration.[1] The form of the nth mode is $E_y = E_n(x) \exp(-j\beta_n z)$

Manuscript received November 25, 1974; revised March 12, 1975. This work was undertaken at Bell Laboratories, Murray Hill, N. J., where the author was on leave from the University of Colorado, Boulder.

The author is with the Department of Electrical Engineering, University of Colorado, Boulder, Colo. 80302.

[1] Examination of properties on the assumption of single mode excitation is permissible if superposition applies; i.e., if the system is linear. This analysis is therefore suitable to the dielectric slab used as a light guide, but holds only approximately for a laser for which nonlinear dynamic effects and saturation result in mode selection and other features not within the scope of the present paper.

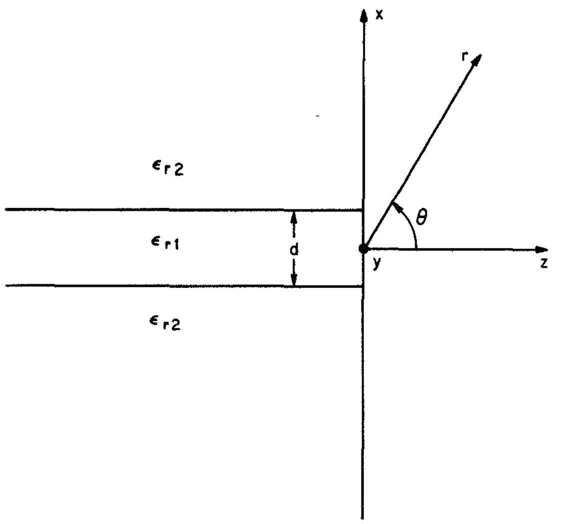

Fig. 1. Laser–air interface.

where x and z are the transverse and axial coordinates, and β_n is the axial propagation coefficient, all normalized to the free-space wavenumber k_0. We will take $E_n(x)$ to be normalized such that

$$\int_{-\infty}^{\infty} E_n^2(x)\, dx = 2\pi.$$

There exist a finite number N of discrete modes together with a continuum of radiating modes. It will not be necessary to distinguish them in this section and as a convenience we will use the symbol $\sum_0^\infty$ to indicate a summation from 0 to N plus an integration over the continuous modal set. In practice N may be quite small, and is often limited to the zero-order mode only. For the case of unit incident mode at the laser–air interface the electric field in the laser is accordingly given by

$$E_y(x) = E_0(x) \exp(-j\beta_0 z) + \sum_0^\infty R_n E_n(x) \exp(j\beta_n z), \quad z < 0 \qquad (1)$$

where the R_n are the mode reflection and conversion coefficients.

The magnetic field component H_x is proportional to $j\partial E_y/\partial z$. At the interface $z = 0$ the forms taken are

$$E_y = E_0(x) + \sum_0^\infty R_n E_n(x) \qquad (2)$$

$$j\partial E_y/\partial z = \beta_0 E_0(x) - \sum_0^\infty R_n \beta_n E_n(x). \qquad (3)$$

It is convenient to put these equations into spectral form. If μ is the spectral variable we take

$$E_n(x) = \int_{-\infty}^{\infty} e_n(\mu) \exp(-jx\mu)\, d\mu$$

$$e_n(\mu) = (1/2\pi) \int_{-\infty}^{\infty} E_n(x) \exp(jx\mu)\, dx \qquad (4)$$

with $e_n(\mu)$ the spectral distribution of the nth mode at the interface. With the normalization assumed for $E_n(x)$ we get

$$\int_{-\infty}^{\infty} e_m(-\mu) e_n(\mu)\, d\mu = \delta_{mn}. \qquad (5)$$

III. THE RADIATED FIELD AND BOUNDARY CONDITIONS

To the right of the laser–air interface the electric field is given by a spectral distribution $T(\mu)$

$$E_y(x,z) = \int_{-\infty}^{\infty} T(\mu) \exp\{-j[\mu x + (1-\mu^2)^{1/2} z]\}\, d\mu, \quad z > 0. \qquad (6)$$

Continuity of tangential electric and magnetic fields at $z = 0$ then gives

$$T(\mu) = e_0(\mu) + \sum_0^\infty R_n e_n(\mu) \qquad (7)$$

and

$$T(\mu)(1-\mu^2)^{1/2} = \beta_0 e_0(\mu) - \sum_0^\infty R_n \beta_n e_n(\mu). \qquad (8)$$

The electric field for $z > 0$ is given by (6). In the far field we take $x = r \sin\theta$, $z = r\cos\theta$, and evaluate the integral by the stationary phase method. This gives

$$E_y(\theta) = C \cos\theta T(\sin\theta) \qquad (9)$$

where C contains the phase and distance factors.

If we put $\mu = \sin\theta$ and write $F(\mu) = (1-\mu^2)^{1/2} T(\mu)$, then $F(\mu)$, from (9), gives the radiation pattern directly. Clearly, this radiation pattern is given by (8); or, alternatively, (7) multiplied by $(1-\mu^2)^{1/2}$. The requirement that, in an *exact* analysis, these two results must be identical, in principle determines the R_n and hence $F(\mu)$. If approximations are made, one formula may yield a much better approximation than the other; and we shall be concerned to extract from the rigorous equations the best results possible for a given order of approximation. Compatability between different approximations is used as a criterion of the accuracy of the results.

IV. SOLUTION FOR $F(\mu)$

Using the orthogonality relations (5) on (8) we get, on putting $\delta_m = 0$, $\delta_0 = 1$,

$$R_m = \delta_m - (1/\beta_m) \int_{-\infty}^{\infty} F(\mu) e_m(-\mu)\, d\mu. \qquad (10)$$

On inserting this into (7) we find

$$F(\mu) = (1-\mu^2)^{1/2} \{2e_0(\mu) - \int_{-\infty}^{\infty} F(\nu) G(\mu,\nu)\, d\nu\} \qquad (11)$$

where

$$G(\mu,\nu) = \sum_0^\infty e_m(\mu)e_m(-\nu)/\beta_m$$

is directly related to the double spectral distribution of the Green's function, as discussed in Appendix I. Equation (11) is an integral equation from which, in principle, $F(\mu)$ could be found; for example by iteration with a constant multiple of $(1-\mu^2)^{1/2}e_0(\mu)$ as a starting function. An alternative is to insert

$$F(\mu) = (1-\mu^2)^{1/2}\{e_0(\mu) + \sum_0^\infty R_n e_n(\mu)\}$$

into (10) and to solve the resultant set of infinite linear equations for R_n by truncation and numerical analysis. This approach was used by Ikegami [7], who retained up to six modes in the computations. His major findings were that mode conversion was less than about 0.5 percent and that R_0, particularly for lasers with a small fractional difference of refractive index, was not far from, but greater than, its value $(n-1)/(n+1)$ for normal incidence. Here $n = \epsilon_r^{1/2}$ is taken as 3.6, or close thereto, for currently used laser materials. A quite different approach is through manipulation of (7) and (8), followed by approximations in which all, or most, of the reflection terms R_n are neglected. It might appear that this is too crude to lead to accurate results, but in fact, surprisingly good results can be obtained in this way. For example, if (7) be multiplied by $(\epsilon_r - \mu^2)^{1/2}$ and the result added to (8), and if we neglect *all* the R_n terms, we get Hockham's formula

$$F(\mu) = \frac{(1-\mu^2)^{1/2}e_0(\mu)[\beta_0 + (\epsilon_r-\mu^2)^{1/2}]}{(\epsilon_r-\mu^2)^{1/2} + (1-\mu^2)^{1/2}}. \quad (12)$$

The modified version comes by going through the same process but retaining just the R_0 term

$$F(\mu) = (1+R_0)\frac{(1-\mu^2)^{1/2}e_0(\mu)[(\epsilon_r-\mu^2)^{1/2} + \beta_0(1-R_0)/(1+R_0)]}{(\epsilon_r-\mu^2)^{1/2} + (1-\mu^2)^{1/2}}. \quad (13)$$

Both these forms are known to give excellent fits to experimental data. In the preceding two cases, the use of the multiplying factor $(\epsilon_r-\mu^2)^{1/2}$ appears to be arbitrary. Its choice is justified solely by the fact that it leads to known results based on the obliquity factor calculation, or on Hockham's more sophisticated analysis. The question is: is there any way of choosing an optimum multiplier such that the best possible result ensues with a neglect of R_n terms beyond a specified point? It turns out that there is such a way of proceeding, and that $(\epsilon_r-\mu^2)^{1/2}$ is indeed the approximate form for the case in which all the R_n are neglected.

V. DETERMINATION OF THE OPTIMUM MULTIPLIER

If we use a factor M, a function of μ, instead of $(\epsilon_r-\mu^2)^{1/2}$, and retain all terms we get an exact formula which is a sort of weighted average of (7) and (8)

$$F(\mu) = \frac{(1-\mu^2)^{1/2}}{M + (1-\mu^2)^{1/2}}\{e_0(\mu)(M+\beta_0) + \Delta\} \quad (14)$$

where

$$\Delta = M\sum_0^\infty R_n e_n(\mu) - \sum_0^\infty R_n\beta_n e_n(\mu). \quad (15)$$

We can make $\Delta = 0$ by choosing M to be given by the equation

$$M = \sum_0^\infty R_n\beta_n e_n(\mu)/\sum_0^\infty R_n e_n(\mu) \quad (16)$$

and if we *knew* the value of the expression on the right of (16) we would, of course, have an exact solution to the problem. All the mode-conversion and reflection effects are contained in M. The solution we can obtain is related to various approximations to (16). We note that, from (3) and (8), the numerator in (16) is related to the spectral component of the transverse component of the reflected magnetic field, and that the denominator is similarly related to the reflected electric field. If we make the *plane-wave assumption* that the incident field *spectral component* is reflected at $z = 0$ as it would be at a uniform dielectric interface, then we can put, for the total reflected field,

$$e(\mu)_{\text{refl}} = R(\mu)e_0(\mu)$$
$$h(\mu)_{\text{refl}} = \epsilon_r^{1/2}\cos\theta' e(\mu)_{\text{refl}}. \quad (17)$$

Here, θ' is the angle of propagation in the dielectric corresponding to μ, and $R(\mu)$ is the reflection in the dielectric at this angle, given by

$$R(\mu) = \frac{\cos\theta' - (1/\epsilon_r - \sin^2\theta')^{1/2}}{\cos\theta' + (1/\epsilon_r - \sin^2\theta')^{1/2}}. \quad (18)$$

Since $\mu = \sin\theta$ is related to θ' by Snell's law of refraction

$$\epsilon_r^{1/2}\sin\theta' = \sin\theta \quad (19)$$

these results give

$$R(\mu) = \frac{(\epsilon_r-\mu^2)^{1/2} - (1-\mu^2)^{1/2}}{(\epsilon_r-\mu^2)^{1/2} + (1-\mu^2)^{1/2}} \quad (20)$$

and

$$\epsilon_r^{1/2}\cos\theta' = (\epsilon_r-\mu^2)^{1/2}. \quad (21)$$

Hence (16) gives, to this approximation, $M = (\epsilon_r-\mu^2)^{1/2}$, and Hockham's formula follows.

In order to put this approach on a systematic basis, and to try to improve on the plane-wave assumption, we assume that the first L modes are retained in their correct form in (14) and that M is chosen to cancel exactly the remaining terms. Then

$$F(\mu) = \frac{(1-\mu^2)^{1/2}}{M+(1-\mu^2)^{1/2}}\left\{e_0(\mu)(\beta_0+M) + \sum_0^L R_m e_m(\mu)(M-\beta_m)\right\} \quad (22)$$

with

$$M = \sum_{L+1}^{\infty} R_n\beta_n e_n(\mu) \Big/ \sum_{L+1}^{\infty} R_n e_n(\mu). \quad (23)$$

In particular, if we take $L = N$ so as to correspond to all the discrete modes only, then the summations in (23) are really integrals over the radiating modes. Since these are very nearly plane waves, we see that M in (23) must be very close to the value $(\epsilon_r - \mu)^{1/2}$ for plane-wave reflection. Hence (22), with this value of M, should be very accurate. Of course, with many modes retained, both discrete and continuous, (22) is correct *whatever* the value of M, since (22) is an arbitrarily weighted average of (7) and (8), which are exact relations in the limit of L approaching infinity. But (22) with $M = (\epsilon_r - \mu^2)^{1/2}$ should be very close in any case whatever the value of L. We can now recognize that the various terms in (22) come from the Huygens' obliquity factors appropriate to field components with propagation factors $-\beta_m$; and providing the R_m are known, (22) is the solution required. If we retain solely the R_0 term we thus get, in a rigorous way, the reflection–modification of Hockham's formula [4]; numerically, the effect of this modification is extremely small, though it gives a very slight additional sharpening of the beam.

VI. ITERATIVE SOLUTION

As a practical matter, approximations based on (7) can be expected to be superior to those based on (8), since the latter converges more slowly because β_m increases with m. Thus two terms only would give, respectively,

$$F(\mu) \approx (1-\mu^2)^{1/2}e_0(\mu)(1+R_0) \quad (24)$$

$$F(\mu) \approx \beta_0 e_0(\mu)(1-R_0) \quad (25)$$

and with the known vanishing of $F(\mu)$ at $\mu = 1$, the first form is clearly much more accurate. The weighted averages (12) and (13) are closer still, and are suitable starting points for an extremely rapid iteration in the integral equation (11). Unfortunately, the integrations involving $G(\mu,\nu)$ cannot be done analytically, so this method is only of use for numerical computations. We show here that the very crude first approximation (25) leads to the reflection-modified form of Hockham's formula after one iteration, using the uniform-dielectric approximation

$$G(\mu,\nu) \approx (\epsilon_r - \mu^2)^{-1/2}\delta(\mu-\nu) \quad (26)$$

found for the Green's function transform in Appendix I.

Putting

$$F(\mu) = \beta_0 e_0(\mu)(1-R_0) + \Delta(\mu)$$

where $\Delta(\mu)$ is the difference between the exact form and the approximation (25), into the right-hand side of the integral equation (11) gives

$$F(\mu) = (1-\mu^2)^{1/2}\left\{2e_0(\mu) - e_0(\mu)(1-R_0) - \int_{-\infty}^{\infty}\Delta(\nu)G(\mu,\nu)\,d\nu\right\}. \quad (27)$$

Here we have made use of the orthogonality of the $e_n(\mu)$ and the normalization (5). If we put $G(\mu,\nu) = (\epsilon_r - \mu^2)^{-1/2}\delta(\mu-\nu) + \Delta G$, where ΔG is the difference between $G(\mu,\nu)$ and its uniform-dielectric approximation (26), then (27) becomes, on replacing $\Delta(\nu)$ by $F(\nu) - \beta_0 e_0(\nu)(1-R_0)$,

$$F(\mu) = (1-\mu^2)^{1/2}\left\{e_0(\mu)(1+R_0) - (\epsilon_r-\mu^2)^{-1/2}[F(\mu) - \beta_0 e_0(\mu)(1-R_0)] - \int_{-\infty}^{\infty}\Delta(\nu)\Delta G\,d\nu\right\}. \quad (28)$$

So far the analysis has been rigorous. The approximation comes by neglecting the supposedly small term in $\Delta(\nu)\Delta G$. The result is the modified form of Hockham's formula, (13), our most accurate equation so far. We would prefer to go through the same process but starting with (24) instead of (25), but $(1-\mu^2)^{1/2}e_0(\mu)$ is not orthogonal to $e_n(\mu)$. Accordingly, we are left with an integration involving $G(\mu,\nu)$ which does not appear to be analytically tractable: though, as already mentioned, the formula should yield accurate numerical results.

The neglected term in (28) depends on $\Delta(\mu)$, the contribution due to higher order discrete modes, and the continuous modes. The latter have spectral components that are mainly concentrated in the region $\mu \gg 0$. For those spectral components with $\mu > 1$ the stationary phase calculation from (6) gives in any case no radiation into the space $z > 0$. Hence the main modification to (13), as depicted in (22), comes from those discrete modes, if any, which are generated at the laser–air interface. If there are none, then (13) is the best simple analytic form we have so far. If, say, the $m = 1$ discrete mode exists, then it should be retained in (22). If the laser is symmetrical about $x = 0$, the first discrete mode to be retained has to be symmetrical also, since otherwise its R_m would be zero. We are accordingly left with the problem of determining the mode-conversion coefficients R_m, as well as the dominant-mode reflection R_0.

VII. CALCULATION OF THE R_m

Applying the orthogonality relations to (7) and (8), with $F(\mu) = (1-\mu^2)^{1/2}T(\mu)$ supposedly known, gives two different formulas for R_m

$$R_m = \int_{-\infty}^{\infty} e_m(-\mu)F(\mu)(1-\mu^2)^{-1/2}\,d\mu \quad (29)$$

$$= -(1/\beta_m) \int_{-\infty}^{\infty} e_m(-\mu) F(\mu)\, d\mu, \qquad m > 0. \quad (30)$$

If we knew the exact form for $F(\mu)$, (29) and (30) would give identical results. With only an approximation, such as (13) or (22), the results cannot be identical, though with a good approximate form for $F(\mu)$ the results should be close. The compatability of (29) and (30) is therefore a measure of the reliability of the formulation. (This statement ignores the exceptional case of an accidental equality of two results, both of which could be wrong.)

When $m = 0$ an additional term, corresponding to the exciting mode, is required, and the equations become

$$1 + R_0 = \int_{-\infty}^{\infty} e_0(-\mu) F(\mu) (1 - \mu^2)^{-1/2}\, d\mu \quad (31)$$

$$\beta_0(1 - R_0) = \int_{-\infty}^{\infty} e_0(-\mu) F(\mu)\, d\mu. \quad (32)$$

We shall examine first the approximation that applies when no discrete mode other than the dominant mode exists. Hence $L = 0$ in (22), and this leads to the reflection-modified form of Hockham's formula (13) for $F(\mu)$. Hence from (31) and (32),

$$1 + R_0 = \int_{-\infty}^{\infty} e_0(\mu) e_0(-\mu) \cdot \frac{[(1 + R_0)M + \beta_0(1 - R_0)]}{M + \Gamma}\, d\mu \quad (33)$$

$$\beta_0(1 - R_0) = \int_{-\infty}^{\infty} e_0(\mu) e_0(-\mu) (1 - \mu^2)^{1/2} \cdot \frac{[(1 + R_0)M + \beta_0(1 - R_0)]}{M + \Gamma}\, d\mu \quad (34)$$

where

$$M = (\epsilon_r - \mu^2)^{1/2}, \qquad \Gamma = (1 - \mu^2)^{1/2}.$$

The question of what value should be used for ϵ_r is perhaps relevant here. Since the derivation is ultimately dependent on a consideration of the plane-wave reflection, from the behavior of the radiating waves at the interface, or alternatively, the approximation (26), it is apparent that it is the bulk medium outside the active material that mainly determines these properties. If we use constants $\epsilon_{r1} = n_1^2$ and $\epsilon_{r2} = n_2^2$, with $n_1 > n_2$, for the materials, respectively, inside and outside the active region, then $\epsilon_r = \epsilon_{r2} = n_2^2$ is the relevant parameter to be used in (33) and (34).

Clearly, these two relations cannot, in general, give identical results. Now the spectral form for $e_0(\mu)$ is mainly concentrated around values associated with near-axial propagation. A very crude approximation is therefore to take $e_0(\mu) = \delta(\mu)$; whence (33) and (34) give

$$\begin{aligned} 1 + R_0 &= [n_2(1 + R_0) + \beta_0(1 - R_0)]/(n_2 + 1) \\ \beta_0(1 - R_0) &= [n_2(1 + R_0) + \beta_0(1 - R_0)]/(n_2 + 1). \end{aligned} \quad (35)$$

It happens that both these relations give the same formula

$$(1 + R_0)/(1 - R_0) = \beta_0. \quad (36)$$

Since [8] $n_2 < \beta_0 < n_1$, (36) determines a value for R_0 approximately equal to that for reflection at normal incidence, i.e., $R_0 = (n - 1)/(n + 1)$ for some average value of n. A slightly different result ensues if we take $\beta_0 = n_1 \cos \alpha$, with α small, and assume $e_0(\mu) = \frac{1}{2}\delta(\mu - \alpha) + \frac{1}{2}\delta(\mu + \alpha)$. Then $(1 + R_0)/(1 - R_0) = \beta_0/\cos \alpha = n_1$, substantially equivalent to (36) for $n_1 \approx n_2$. Note that this is not at all the same as the value calculated from plane-wave incidence at angle α, particularly for large n_1. The real feature that is relevant here is that it is mainly small values of μ which are important in the integrals in (33) and (34). If we expand M and Γ around $\mu = 0$, retain terms up to μ^2, integrate and multiply both sides by $(n_2 + 1)$, then we get, to first order,

$$\begin{aligned} (n_2 + 1)(1 + R_0) &= n_2(1 + R_0) + \beta_0(1 - R_0) \\ &\quad + (\Delta_0/2n_2)[(n_2 - 1)(1 + R_0) \\ &\quad + \beta_0(1 - R_0)] \\ (n_2 + 1)\beta_0(1 - R_0) &= n_2(1 + R_0) + \beta_0(1 - R_0) \\ &\quad - (\Delta_0/2n_2)[(n_2^2 - n_2 + 1) \\ &\quad \cdot (1 + R_0) + \beta_0(1 - R_0)(n_2 - 1)] \end{aligned} \quad (37)$$

where

$$\Delta_m = \int_{-\infty}^{\infty} e_0(\mu) e_m(-\mu) \mu^2\, d\mu. \quad (38)$$

To the extent that $e_0(\mu)$ is concentrated mainly around $\mu = 0$, (38) indicates that Δ_0 is small. Equations (37) then give expressions for $(1 + R_0)/(1 - R_0)$ which, although no longer identical, possess the following expansions for small Δ_0:

$$\frac{1 + R_0}{1 - R_0} \approx \beta_0[1 + \Delta_0/2 + \Delta_0^2(n_2 - 1)/4n_2]$$

$$\frac{1 + R_0}{1 - R_0} \approx \beta_0[1 + \Delta_0/2 + \Delta_0^2(n_2 - 1 + 1/n_2)/4n_2]. \quad (39)$$

The difference between the two is the very small term $\beta_0 \Delta_0^2/4n_2^2$ and this seems a fair measure of the accuracy of these formulas.

The calculation of Δ_m is considered in Appendix II. Since, from (38), Δ_0 is, in any case, a positive quantity, (39) shows that R_0 is always greater than the value given by (36). For small $n_1^2 - n_2^2 = \Delta\epsilon_r$ we get, for the case of a single symmetrical refractive index step

$$\frac{1+R_0}{1-R_0} \approx n_1\left\{1+\tfrac{1}{2}\Delta\epsilon_r\cos^2\varphi\left[1-\frac{1}{1+\varphi\tan\varphi}-\frac{1}{\epsilon_{r1}}\right]\right\} \tag{40}$$

where φ is the smallest solution of $\varphi = (\pi d/\lambda)(\Delta\epsilon_r)^{1/2}\cos\varphi$ and d is the (nonnormalized) active layer thickness. Numerically, this formula seems to be in excellent accord with Ikegami's curves [7]. For example, as a somewhat extreme case, if we take $\epsilon_{r1} = 12.9$, $\epsilon_{r2} = 11.1$, so that $(n_1 - n_2)/n_1 = 0.07$, then Ikegami's results for $d = 0.3\mu$, $\lambda = 0.86\mu$ give $R_0 = 0.628$ while (40) gives 0.618. The normal incidence figure on a dielectric–air interface with $\epsilon_r = 12.9$, $(n = 3.6)$, is, by contrast, 0.565; and for $\epsilon_r = 11.1$ $(n = 3.33)$, it is 0.538.

For $(\pi d/\lambda)(\Delta\epsilon_r)^{1/2}$ small enough it would be possible for $\varphi\tan\varphi$ to be less than $1/(\epsilon_{r1}-1)$. Should this happen, the multiplier of $\Delta\epsilon_r$ in (40) could go negative, and R_0 would dip below the value corresponding to n_1. This can only happen for very small d/λ and/or $\Delta\epsilon_r$ and is outside the range of Ikegami's curves. All of these seem to rise from near the value determined by n_1, though for $d = 0.2\mu$ the behavior of the curve near the origin is somewhat different. The region in which this effect can occur is approximately determined by

$$(\lambda/\pi d)^2 > (\epsilon_{r1}-1)\Delta\epsilon_r \tag{41}$$

and with Ikegami's values for λ and ϵ it would begin to occur when $d < 0.155\mu$ and $(n_1-n_2)/n_1 < 0.01$. However, $\Delta\epsilon_r$ would necessarily be small in this range, so that the decrease of R_0 below the value corresponding to n_1 at normal incidence would, with current materials, seem to be negligible.

To calculate R_m for $m > 0$ we return to (29) and (30) with $F(\mu)$ given by (22). We shall illustrate the method by evaluating R_1 and its effect on the value of R_0. Following the process that led to (33) and (34) but with $F(\mu)$ taken as

$$F(\mu) = \frac{(1-\mu^2)^{1/2}}{M+\Gamma}\left[\{M(1+R_0)+\beta_0(1-R_0)\}e_0(\mu) + R_1(M-\beta_1)e_1(\mu)\right] \tag{42}$$

we get the two sets of equations

$$1+R_0 = \int_{-\infty}^{\infty} e_0(\mu)e_0(-\mu)f_1(\mu)\,d\mu + \int_{-\infty}^{\infty} e_1(\mu)e_0(-\mu)f_2(\mu)\,d\mu \tag{43}$$

$$\beta_0(1-R_0) = \int_{-\infty}^{\infty} e_0(\mu)e_0(-\mu)f_1(\mu)\Gamma\,d\mu + \int_{-\infty}^{\infty} e_1(\mu)e_0(-\mu)f_2(\mu)\Gamma\,d\mu \tag{44}$$

and

$$R_1 = \int_{-\infty}^{\infty} e_1(\mu)e_1(-\mu)f_2(\mu)\,d\mu + \int_{-\infty}^{\infty} e_0(\mu)e_1(-\mu)f_1(\mu)\,d\mu \tag{45}$$

$$-\beta_1 R_1 = \int_{-\infty}^{\infty} e_1(\mu)e_1(-\mu)f_2(\mu)\Gamma\,d\mu + \int_{-\infty}^{\infty} e_0(\mu)e_1(-\mu)f_1(\mu)\Gamma\,d\mu. \tag{46}$$

As before, $\Gamma = (1-\mu^2)^{1/2}$, $M = (\epsilon_r - \mu^2)^{1/2}$, and here we have put

$$f_1(\mu) = [M(1+R_0)+\beta_0(1-R_0)]/(M+\Gamma) \tag{47}$$

$$f_2(\mu) = R_1(M-\beta_1)/(M+\Gamma). \tag{48}$$

Equations (43) and (44) are the same as (33) and (34), except for the terms in $f_2(\mu)$. As explained earlier, much of the contribution in these integrals arises from values of μ near 0. Since, as will be apparent shortly, R_1 is of order Δ_1, which is small, while the factor $M - \beta_1$ in $f_2(\mu)$ is also small for small μ, the additional terms is (43) and (44) are of second order. Hence the first-order calculations of R_0 persist unaltered. Turning our attention to the pair (45) and (46) we replace f_1, f_2, and Γ by their small-μ approximations, to give, after the manner of getting (37),

$$R_1(1+n_2) = R_1(n_2-\beta_1) + (\Delta_1/2n_2)[(n_2-1)(1+R_0) + \beta_0(1-R_0)] \tag{49}$$

$$-\beta_1R_1(1+n_2) = R_1(n_2-\beta_1) - (\Delta_1/2n_2)[(n_2^2-n_2+1)\cdot(1+R_0) + \beta_0(1-R_0)(n_2-1)]. \tag{50}$$

These two relations are compatible apart from second-order terms, and give

$$R_1 \approx \frac{\Delta_1}{2n_2(1+\beta_1)}[(n_2-1)(1+R_0)+\beta_0(1-R_0)] \tag{51}$$

where Δ_1 is given by (38) with $m = 1$. It is evaluated in Appendix II.

Since $\beta_1 \approx n$ and $\beta_0(1-R_0)/(1+R_0) \approx 1$ to first order, (51) can be approximated by the simpler

$$R_1 \approx \Delta_1 n/(1+n)^2 \tag{52}$$

for some average value of n between n_1 and n_2. Clearly, to this order, the same calculation holds for all the discrete R_m to give

$$R_m \approx \Delta_m n/(1+n)^2, \qquad 0 < m \leq N. \tag{53}$$

The method does not apply to the leaky modes because, for them, the restriction $n_2 < \beta_m < n_1$ does not hold, and the approximations contingent on this are no longer valid. We can still use (29) or (30) with a suitable approximation for $F(\mu)$ but the effective values of μ are no longer restricted to those around zero. Only a numerical integration seems possible for evaluation.

VIII. CONCLUSIONS

A formulation of the laser field has been achieved and examined from the point of view of several different approximate solutions, Hockham's formula implicitly takes the radiating modes into account and is expected to be quite accurate. The reflection–modification to take the dominant mode reflection explicitly into consideration gives a formal improvement, though its numerical effect is slight. Similarly, if higher order discrete modes can occur, (42) or the more general (22) can be used. These terms follow the Huygens' obliquity factor analysis of Lewin [4], though they are here produced in a more rigorous manner. The mode-conversion coefficients that appear can be calculated analytically, and several useful approximate forms are given for the case of a small refractive index difference. The basic method would appear to have a quite general field of application.

APPENDIX I
THE GREEN'S FUNCTION

The Green's function in cylindrical coordinates represents the electric field from a unit line source. A general representation of the fields of a dielectric slab from a source located at $z = 0$ is

$$E_y = \sum_0^\infty E_m(x) A_m \exp(-j\beta_m |z|) \qquad \text{(A1)}$$

$$-\xi_0 H_x = \sum_0^\infty E_m(x) A_m \beta_m \exp(-j\beta_m |z|)\, \mathrm{sgn}(z) \qquad \text{(A2)}$$

where $\xi_0 = (\mu_0/\epsilon_0)^{1/2}$ and $\mathrm{sgn}(z) = (\pm 1)$ according to the sign of z. If we take the source to be of strength $(-1/\zeta_0)$ and to be a delta function $\delta(x - x')$ located at $x = x'$ then the requirement that the discontinuity in H_x should equal the current at $z = 0$ gives

$$\sum_0^\infty E_m(x) 2A_m\beta_m = \delta(x - x'). \qquad \text{(A3)}$$

Noting that the E_m are mutually orthogonal and have been normalized to 2π (see (5)) we get

$$A_m = E_m(x')/4\pi\beta_m. \qquad \text{(A4)}$$

Hence

$$g(x,x') = \frac{1}{4\pi} \sum_0^\infty E_m(x) E_m(x')/\beta_m \qquad \text{(A5)}$$

and is $(-1/\xi_0)$ times the Green's function for the laser structure.

Multiplying (A5) by $\exp(jx'\mu)\exp(-jx\nu)$, where μ and ν are spectral variables, and integrating with respect to x and x' from $-\infty$ to $+\infty$ gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,x') \exp[j(x'\mu - x\nu)]\, dx\, dx' = \pi \sum_0^\infty e_m(\mu) e_m(-\nu)/\beta_m \qquad \text{(A6)}$$

where

$$e_m(\mu) = (1/2\pi) \int_{-\infty}^{\infty} E_m(x) \exp(j\mu x)\, dx.$$

Defining $G(\mu,\nu)$ as the series on the right of (A6) we get

$$G(\mu,\nu) = \frac{1}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,x') \exp[j(x'\mu - x\nu)]\, dx\, dx'. \qquad \text{(A7)}$$

Now in the limiting case of a uniform dielectric of refractive index n_2 the radiation from a unit source is readily found to be proportional to the Hankel function $H_0^{(2)}(n_2\rho)$ where ρ is the normalized radial coordinate. With the strength of source used here the proportionality constant can be shown to be $(\frac{1}{4})$. Since at $z = 0$ we have $\rho = |x - x'|$ we get

$$g(x,x') = (\tfrac{1}{4}) H_0^{(2)}(n_2 |x - x'|). \qquad \text{(A8)}$$

Substituting in (A7) and using the integral

$$H_0^{(2)}(n|\zeta|) = (1/\pi) \int_{-\infty}^{\infty} \exp(j\zeta t)(n^2 - t^2)^{-1/2}\, dt \qquad \text{(A9)}$$

gives

$$G(\mu,\nu) = (1/4\pi^2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (n^2 - t^2)^{-1/2} \cdot \exp\{j[xt - x't + x'\mu - x\nu]\}\, dx\, dx'\, dt. \qquad \text{(A10)}$$

Now the expression $\int_{-\infty}^{\infty} \exp(jx\tau)\, dx$ is zero except when $\tau = 0$, when it is infinite. By integrating with respect to τ over a short interval surrounding $\tau = 0$ we get the value 2π. Hence the integral can be represented by $2\pi\delta(\tau)$. Using this result to evaluate the x and x' integrals in (A10) we get

$$G(\mu,\nu) = \int_{-\infty}^{\infty} (n^2 - t^2)^{-1/2} \delta(t - \nu)\delta(t - \mu)\, dt. \qquad \text{(A11)}$$

This expression is zero unless $\mu = \nu$, when it is ∞, i.e., it is a delta function of the $(\mu - \nu)$. Integrating with respect to μ from $-\infty$ to $+\infty$ we get

$$\int_{-\infty}^{\infty} G(\mu,\nu)\, d\mu = \int_{-\infty}^{\infty} (n^2 - t^2)^{-1/2}\delta(t - \nu)\, dt = (n^2 - \nu^2)^{-1/2}.$$

Hence

$$G(\mu,\nu) = (n^2 - \nu^2)^{-1/2}\delta(\mu - \nu). \qquad \text{(A12)}$$

Equation (A12) is the limiting form of $G(\mu,\nu)$ as $\epsilon_1 \rightarrow \epsilon_2$.

The same approach enables a form for $G(\mu,\nu)$ to be found when the laser structure is specified. As an example we give briefly the case of a uniform symmetric laser with $\epsilon_r = n_1^2$, $-d/2 < x < d/2$, $\epsilon_r = n_2^2$, $|x| > d/2$, as shown in Fig. 2. Only the even modes for which $e_m(\mu) = e_{m-}(\mu)$ will survive if the feeding is chosen to be symmetrical, i.e., a pair of sources located at $\pm x'$.

The analysis for this slab geometry is given by Marcuse [8]. The form taken by the field depends on whether $|x|$ is greater or smaller than $d/2$, and the contribution from the pair of sources at $\pm x'$ depends equally on whether they are inside or outside the slab. We will indicate by the notation $\{f_1(x), f_2(x)\}$ a function of x which takes the form f_1 when $|x| < d/2$ and f_2 when $|x| > d/2$. We shall consider integration over an infinite range of a *real* spectral variable σ so that the Fourier representation is complete. It turns out that reciprocity requirements (symmetry in x and x') have the effect of *excluding* from the integral those values of σ for which discrete modes can occur. Hence they have to be allowed for explicitly: what this means is that the analysis produces the leaky mode components and that the discrete modes have to be added to produce the complete solution. The analysis yields the following forms for the electric field:

1)

$$|x'| < d/2,\ 4\pi E_y = \sum_0^N E_m(x)E_m(x') \exp(-j\beta_m|z|) + \int_{-\infty}^{\infty} (n_2^2 - \sigma^2)^{-1/2} A_\sigma \cos(\sigma' x') \cdot \{A_\sigma \cos(\sigma' x), \cos(\Theta_\sigma - \sigma|x|)\} \cdot \exp[-j(n_2^2 - \sigma^2)^{1/2}|z|]\, d\sigma \quad \text{(A13)}$$

2)

$$|x'| > d/2,\ 4\pi E_y = \sum_0^N E_m(x)E_m(x') \exp(-j\beta_m|z|) + \int_{-\infty}^{\infty} (n_2^2 - \sigma^2)^{-1/2} \cdot \cos(\Theta_\sigma - \sigma|x'|) \cdot \{A_\sigma \cos(\sigma' x), \cos(\Theta_\sigma - \sigma|x|)\} \cdot \exp[-j(n_2^2 - \sigma^2)^{1/2}|z|]\, d\sigma. \quad \text{(A14)}$$

Herein we have defined

$$\sigma' = (\sigma^2 + \Delta\epsilon_r)^{1/2}$$

$$A_\sigma = [1 + \Delta\epsilon_r \sin^2(\tfrac{1}{2}d\sigma')/\sigma^2]^{-1/2}$$

$$\Theta_\sigma = \sin^{-1}\{[\sin\tfrac{1}{2}d\sigma \cos\tfrac{1}{2}d\sigma' - (\sigma'/\sigma)\sin\tfrac{1}{2}d\sigma' \cos\tfrac{1}{2}d\sigma]A_\sigma\}. \quad \text{(A15)}$$

It can now be verified that

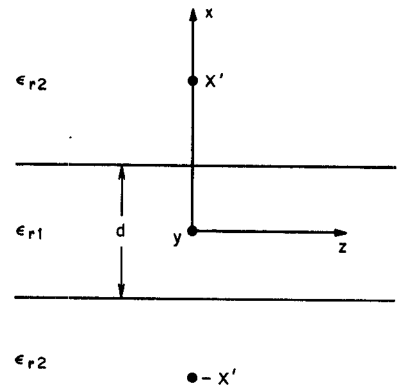

Fig. 2. Slab geometry, with line sources at $\pm x'$.

1) (A13) and (A14) are each solutions of the wave equation.

2) They and their first derivatives are continuous at $x = \pm d/2$.

3) They are symmetrical in x and x'.

4) For x and x' *large* the dominant contribution comes from two line sources, each of half strength, at $\pm x'$ in medium 2, the remainder being effects attributable to reflections and discrete modes at the slab region.

5) The representations are complete in both media. Hence (A13) and (A14) are a valid form for the solution for the Green's function for the problem. In this, the symmetrical-fed case, only symmetrical modes are retained in the summation in (A13) and (A14). The more general problem can be tackled by the same method by including the unsymmetrical slab solution in the form for the fields.

The verification of points 1), 2), and 3) above is straightforward. Point 5) is obvious by inspection for $|x| > d/2$ since σ covers all real values. For $|x| < d/2$ the variable $\sigma' = (\sigma^2 + \Delta\epsilon_r)^{1/2}$ is needed to ensure the same field variation $\exp[-j(n_2^2 - \sigma^2)^{1/2}z]$ in both media, and, therefore omits, for real σ, the region $-(\Delta\epsilon_r)^{1/2} < \sigma' < (\Delta\epsilon_r)^{1/2}$. This corresponds to the internal reflection range, so that the discrete modes have to be added separately. The reason that the solution takes on this characteristic lies in the effective decoupling of the discrete mode solutions inside the slab from the fields for large x and x'. The solution (A14) is built up on the latter and gives

$$4\pi E_y \sim \tfrac{1}{2}\int_{-\infty}^{\infty} (n_2^2 - \sigma^2)^{-1/2} [\cos(\sigma(x - x')) + \cos(2\Theta_\sigma - \sigma(x + x'))]\, d\sigma \quad \text{(A16)}$$

for $x, x' \gg d/2$, the discrete modes being exponentially attenuated. When x and x' are both large, the second term oscillates rapidly and its integration approaches zero. The first term is integrable exactly, and gives

$$E_y = (\tfrac{1}{8})H_0^{(2)}(n_2|x - x'|) \quad \text{(A17)}$$

as is required for a half-strength source at x' in medium 2. The remaining term in (A16) is at a distance $2x'$ away and corresponds to the effect of the slab on the radiation there due to (A17). This verifies point 4) and completes the demonstration of the correctness of the solution.

The double spectral distribution Green's function comes by multiplying (A13) and (A14) by $\exp[j(x\mu - x'\nu)]$ and integrating with respect to x and x' from $-\infty$ to $+\infty$. This gives

$$G(\mu,\nu) = (1/4\pi^2)\int_{-\infty}^{\infty}(n_2^2 - \sigma^2)^{-1/2}e(\mu,\sigma)e(-\nu,\sigma)\,d\sigma + \sum_0^N e_m(\mu)e_m(-\nu)/\beta_m \quad \text{(A18)}$$

where

$$e(\mu,\sigma) = \pi\cos\Theta_\mu[\delta(\mu+\sigma)+\delta(\mu-\sigma)] + 2\Delta\epsilon_r A_\sigma \frac{\mu\sin\frac{1}{2}d\mu\cos\frac{1}{2}d\sigma' - \sigma'\cos\frac{1}{2}d\mu\sin\frac{1}{2}d\sigma'}{(\mu^2-\sigma^2)(\mu^2-\sigma'^2)}. \quad \text{(A19)}$$

Unfortunately, the simplicity of the form of (A12) has now been largely lost. This actual form can, in fact, be abstracted from the δ-function products coming from the $e(\mu,\nu)e(-\nu,\sigma)$ multiplication in (A18), where it appears along with a factor $\cos\Theta_\mu\cos\Theta_\nu$. The angle Θ_μ is very small except for a range within $\mu^2 < \Delta\epsilon_r$, when it runs to $\pm\pi/2$. At the same time A_μ passes through zero. These rather awkward properties make it difficult to express (A18) in the form (A12) plus a correction term proportional to $\Delta\epsilon_r$, a structure which does seem indicated by this analysis. If this could be done a lot of useful improvements to the earlier formulas could probably be achieved.

APPENDIX II
EVALUATION OF Δ_m

Before proceeding to a computation of Δ_m for the symmetrical slab arrangement, it may be as well to repeat that $e_0(\mu)$ is governed mainly by a range of μ near the origin. For large μ, $e_m(\mu) \sim 0(\mu^{-3})$ so that $\int_{-\infty}^{\infty} e_0(\mu)\cdot e_m(-\mu)\mu^2\,d\mu$ has an integrand that varies as μ^{-4} for large μ. Its rapid convergence is therefore assured.

Using the spectral formula for $e_m(-\mu)$ we get

$$\Delta_m = (1/4\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0(x)E_m(x')\cdot\exp[j\mu(x-x')]\mu^2\,d\mu\,dx\,dx'.$$

This expression can be evaluated as follows. Replace $\mu^2\exp[j\mu(x-x')]$ by $(\partial^2/\partial x^2)\exp[j\mu(x-x')]$ and evaluate the μ integration to give $(\partial^2/\partial x^2)\delta(x-x')$. The integral is now integrated by parts twice with respect to x, and the x' integration performed. This gives

$$\Delta_m = (-1/2\pi)\int_{-\infty}^{\infty} E_m(x)E_0''(x)\,dx = (1/2\pi)\int_{-\infty}^{\infty}[\epsilon_{r2} - \beta_0^2 + \{\Delta\epsilon_r,0\}]E_m(x)E_0(x)\,dx \quad \text{(A20)}$$

on using the wave equation for $E_0(x)$. Equation (A20) is, in fact, rather more general in application if $\{\Delta\epsilon_r,0\}$ is replaced by $\Delta\epsilon_r(x)$, a varying deviation from the value ϵ_{r2} taken for large x.

For $m = 0$ (A20) gives

$$\Delta_0 = \epsilon_{r2} - \beta_0^2 + (\Delta\epsilon_r/\pi)\int_0^{d/2} N_0^2\cos^2(\sigma x)\,dx \quad \text{(A21)}$$

where N_0 is a normalizing factor and σ the transverse wavenumber for the dominant mode. N_0 is found by the requirement

$$\int_{-\infty}^{\infty} N_0^2\{\cos^2(\sigma x), \cos^2(\sigma d/2)\cdot\exp[-2\gamma(|x| - d/2)]\}\,dx = 2\pi \quad \text{(A22)}$$

where $\gamma^2 = \beta_0^2 - n_2^2$.

Carrying out the indicated integrations and using the continuity condition at $x = d/2$, which can be put in the form

$$\varphi\sin\varphi = \gamma d\cos\varphi \quad \text{(A23)}$$

we get

$$N_0^2 = \frac{(4\pi/d)\varphi\tan\varphi}{1+\varphi\tan\varphi}. \quad \text{(A24)}$$

Here $\varphi = \sigma d/2$, and (A23) can also be written, since $\sigma^2 = \Delta\epsilon_r - \gamma^2$,

$$\varphi = d(\Delta\epsilon_r)^{1/2}\cos\varphi. \quad \text{(A25)}$$

The integrations in (A21) are straightforward. Recalling that, so far, all quantities have been normalized with respect to k_0, the free-space wavenumber, we get (40) of Section VII.

Equation (A25) is a transcendental equation for φ, which determines σ, γ, and β_0. Putting $d(\Delta\epsilon_r)^{1/2} = D$, the equation $\varphi = D\cos\varphi$ clearly has the solution $\varphi \approx D$ for small D and $\varphi \approx \pi/2$ for large D. An approximate relation that holds over the entire range can be built up and expressed in the form

$$\varphi = \frac{D}{(1+D^2)^{1/2}}\left(\frac{1+D^2+0.7765D^4}{1+D^2+0.1098D^4}\right)^{1/4}. \quad \text{(A26)}$$

In particular, the simpler form $\varphi = D/(1+D^2)^{1/2}$ is very useful for $D < 1$.

For values of m other than zero, the orthogonality of the E_m reduces the first part of (A20) to zero and we are left with

$$\Delta_m = (\Delta\epsilon_r/\pi)\int_0^{d/2} N_0 N_m\cos(\sigma_0 x)\cos(\sigma_m x)\,dx \quad \text{(A27)}$$

where the subscripts 0 and m refer to the corresponding roots for σ, via φ, in (A23). The integration gives

$$\Delta_m = \Delta\epsilon_r\left[\frac{\sin(\varphi_0+\varphi_m)}{\varphi_0+\varphi_m} + \frac{\sin(\varphi_0-\varphi_m)}{\varphi_0-\varphi_m}\right]\cdot\left[\frac{\varphi_0\tan\varphi_0\cdot\varphi_m\tan\varphi_m}{(1+\varphi_0\tan\varphi_0)(1+\varphi_m\tan\varphi_m)}\right]^{1/2}. \quad \text{(A28)}$$

REFERENCES

[1] H. C. Casey, M. B. Panish, and J. L. Merz, "Beam divergence of the emission from double-heterostructure injection lasers," *J. Appl. Phys.*, vol. 44, pp. 5470–5475, 1973.
[2] P. A. Kirkby and G. H. B. Thompson, "The effect of double heterostructure waveguide parameters on the far field emission patterns of lasers," *Opto-Electron.*, vol. 4, pp. 323–334, 1972.
[3] G. A. Hockham, "Radiation from a solid-state laser," *Electron. Lett.*, vol. 9, pp. 389–391, 1973.
[4] L. Lewin, "Obliquity-factor for radiation from a solid-state laser," *Electron. Lett.*, vol. 10, pp. 134–135, 1974.
[5] ——, "Obliquity-factor correction to solid-state laser radiation patterns," *J. Appl. Phys.*, May 1975.
[6] J. K. Butler and J. Zoroofchi, "Radiation fields of GaAs–AlGa)As injection lasers," *J. Quantum Electron.*, vol. QE-10, pp. 809–815, Oct. 1974.
[7] T. Ikegami, "Reflectivity of mode at facet and oscillation mode in double-heterostructure injection lasers," *IEEE J. Quantum Electron.*, vol. QE-6, pp. 470–476, June 1972.
[8] D. Marcuse, *Light Transmission Optics.* New York: Van Nostrand-Reinhold, 1972, pp. 305–325.

Short Papers

Characteristic Impedance and Field Patterns of the Shielded Microstrip on a Ferrite Substrate

DAVID T. YEH AND DONALD M. BOLLE, SENIOR MEMBER, IEEE

Abstract—**The dispersion relation, field patterns, and current density at the interface of a shielded microstrip on ferrite substrate while operating at remanence is obtained and the characteristic impedance of such a structure is presented.**

In a paper by Minor and Bolle [1], the dispersion relation of a shielded microstrip on a ferrite substrate transversly magnetized in the plane of the substrate was analyzed. The method of solution used was to construct an appropriate modal expansion in each of the two media. The boundary conditions at the interface were then expressed in terms of two coupled integral equations which were subsequently solved by the method of moments. An estimate of 0.5-percent accuracy using a matrix as small as 5×5 was reported.

In this short paper, we obtain the characteristic impedance based on the theory of [1]. The earlier computer program was modified so as to yield numerical results for the characteristic impedance.

The model of the shielded microstrip is shown in Fig. 1. The waveguide walls and the strip are all presumed perfectly conducting. The strip is infinitely thin, and each of the two regions may be either dielectric- or ferrite-loaded. We define the characteristic impedance of such a structure by (see Fig. 1)

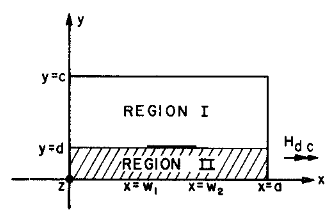

Fig. 1. The shielded microstrip.

$$Z_0 \triangleq \frac{V}{I} \tag{1}$$

where

$$V = -\int_0^d E_{IIy}\left(x = \frac{w_1 + w_2}{2}, y\right) dy \tag{2}$$

and

$$I = \int_{w_1}^{w_2} J_z^E(x)\, dx. \tag{3}$$

E_{IIy} is the y component of the electric field in region II. $J_z^E(x)$ is the axial electric current density. Both of these quantities may be calculated directly once the propagation factor β is obtained for a time dependence of the form $\exp[j\omega t]$. The path of integration taken for the voltage integral is at the midpoint of the strip with $x = (w_1 + w_2)/2$. The current I is the total axial current in the direction of propagation.

To ensure the correctness and establish the accuracy of the program and of the formulation, comparison with previous results

Manuscript received September 25, 1974; revised February 7, 1975. This work was supported by the National Science Foundation under Grant GK-31591.

The authors are with the Division of Engineering, Brown University, Providence, R. I. 02912.